

DYNAMICAL BERCOVICI-PATA BIJECTION: HERMITIAN REPRESENTATION OF FREE LÉVY PROCESSES.

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ABSTRACT. A dynamical Bercovici-Pata bijection between classical and free Lévy processes is established via an ensemble representation of unitarily invariant Hermitian matrix-valued Lévy processes. Furthermore, functional asymptotics of their empirical spectral processes towards free Lévy processes is proved. This result recovers a dynamical version of Wigner's theorem and introduces a dynamical version of Marchenko-Pastur's theorem providing the free Poisson process as the noncommutative limit process.

Key words: Asymptotic spectral distribution, Burger equation, free Brownian motion, free infinitely divisible distribution, free Lévy process, Hermitian Brownian motion, Hermitian Lévy process, interacting particles system, measure-valued process.

1. INTRODUCTION

Let $\{B^{(n)}(t)\}_{t \geq 0} = \{(b_{jk}(t))_{j,k=1}^n\}_{t \geq 0}$ be the $n \times n$ Hermitian matrix-valued Brownian motion where $(b_{jj}(t))_{j=1}^n$, $(\operatorname{Re} b_{jk}(t))_{j < k}$, $(\operatorname{Im} b_{jk}(t))_{j < k}$ is a set of n^2 independent one-dimensional Brownian motions with parameter $\frac{t}{2}(1 + \delta_{jk})$. This matrix-valued process was first considered by Dyson [15] and the study of its eigenvalues process leads to several primary results in Random Matrix Theory (RMT), noncolliding particles, free probability, and laws of noncommutative processes.

First, for any fixed $t > 0$, $B^{(n)}(t)$ is a Gaussian Unitary Ensemble (GUE) of random matrices with parameter t and its matrix distribution is invariant under unitary conjugation as well as infinitely divisible with respect to the classical convolution of matrix distributions, [1], [19]. Let $\{(\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t))\}_{t \geq 0}$ be the n -dimensional stochastic process of the eigenvalues of $B^{(n)}$ and consider the empirical spectral process of the re-scaled matrix $B^{(n)}/\sqrt{n}$

$$(1.1) \quad \mu_t^{(n)} = \frac{1}{n} \sum_{j=1}^n \delta_{\tilde{\lambda}_j(t)}, \quad t \geq 0,$$

where $\tilde{\lambda}_j(t) = \lambda_j(t)/\sqrt{n}$ and δ_x is the unit mass at x .

From the fundamental work of Wigner [31] in RMT, for each fixed $t > 0$, $\mu_t^{(n)}$ converges as n goes to infinity, weakly almost surely, to the semicircle or Wigner distribution with parameter t :

$$(1.2) \quad w_t(dx) = \frac{1}{2\pi t} \sqrt{4t - x^2} 1_{[-2\sqrt{t}, 2\sqrt{t}]}(x) dx;$$

see also [1], [19], [28]. The Wigner distribution w_t is infinitely divisible with respect to the free \boxplus -convolution and it also appears as the limiting distribution in the free central limit theorem [17], [21], [30].

In this sense, w_1 is the free counterpart of the Gaussian distribution in classical infinite divisibility, playing in free probability the role the Gaussian distribution does in classical probability. This is the starting point of the subject of free infinite divisibility, [17], [21], [30]. Moreover, the family $\{w_t\}_{t \geq 0}$ is the law of free Brownian motion, a family of selfadjoint elements $\{Z_t\}_{t \geq 0}$ in a noncommutative probability space that has free increments and is such that for each $0 \leq s \leq t$, $Z_t - Z_s$ has the law w_{t-s} ; see Biane [8].

Second, keeping $n \geq 1$ fixed, in a pioneering work, Dyson [15] realized that the eigenvalue dynamics is described by a diffusion process with non-smooth drift satisfying the Itô Stochastic Differential Equation (SDE)

$$(1.3) \quad d\lambda_i(t) = dW_i(t) + \sum_{j \neq i} \frac{dt}{\lambda_i(t) - \lambda_j(t)}, \quad t \geq 0, 1 \leq i \leq n,$$

where W_1, \dots, W_n are independent one-dimensional standard Brownian motions, see also [1], [28]. The stochastic process $\{(\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t))\}_{t \geq 0}$ is called the Dyson Brownian motion corresponding to the GUE. This is a primary example of a system of interacting particles governed by a SDE with strong interactions due to the non-smooth drift coefficient, a phenomenon associated with the process of eigenvalues of several matrix continuous-time processes; see [10], [11].

Third, a dynamical version of Wigner's theorem is possible: this follows from the study of the limiting laws of measure-valued processes of interacting diffusions with non-smooth drift coefficient, as considered by Rogers and Shi [27]. Namely, the empirical spectral measure-valued processes $\{\{\mu_t^{(n)}\}_{t \geq 0} : n \geq 1\}$ converge to the family of measures $\{w_t\}_{t \geq 0}$. This functional asymptotic takes place in $C(\mathbb{R}_+, \text{Pr}(\mathbb{R}))$, the space of continuous functions from \mathbb{R}_+ into $\text{Pr}(\mathbb{R})$, endowed with the topology of uniform convergence on compact intervals of \mathbb{R}_+ , where $\text{Pr}(\mathbb{R})$ is the space of probability measures on \mathbb{R} endowed with the topology of weak convergence; see also [1], [13]. A similar dynamical asymptotic behavior in the case of other matrix (continuous-time) diffusions has been considered, leading also to free Brownian motion [14], [27] or other noncommutative processes like the dilation of the free Poisson process [13], [25]. The latter is a dynamical version of the Marchenko–Pastur law, but the noncommutative limiting process is not a free Lévy process. Recently, the case of a Hermitian fractional Brownian motion was considered in [23], obtaining the non-commutative law of the fractional Brownian motion introduced in [22].

Lastly, there is a general bijection between the set of classical infinitely divisible distributions and the set of free infinitely divisible distributions, the so-called Bercovici–Pata bijection [6], see also [3], [4]. Benaych–Georges [5] and Cabanal–Duvillard [12] explained this bijection via random matrix models. Their work constitutes a generalization of the Wigner semicircle law for the GUE to more general random matrices. The distributions of these random matrices share similar properties to those of the GUE, such as having an infinitely divisible matrix distribution which is invariant under unitary conjugation (Lévy Unitary Ensemble, LUE).

The goal of this paper is to find a dynamical version of the Bercovici–Pata bijection between classical Lévy processes and free Lévy processes. We are interested in seeking time-varying versions of Wigner's theorem for free Lévy process similar to the known results for the case of free Brownian motion. The latter noncommutative processes were considered in Biane [9]; as a free Brownian motion, they have stationary and free increments and there is a one-to-one correspondence with free infinitely divisible distributions.

More specifically, the main result of this paper is summarized as follows. Let $\mathcal{D}(\mathbb{R}_+, \text{Pr}(\mathbb{R}))$ denote the space of right continuous functions with left limits from \mathbb{R}_+ into $\text{Pr}(\mathbb{R})$, endowed with the Skorohod topology, where

$\Pr(\mathbb{R})$ is the space of probability measures on \mathbb{R} endowed with the topology of weak convergence. Given an $n \times n$ Hermitian process $X_t^{(n)}$ with eigenvalues $\lambda_1^{(n)}(t) \geq \lambda_2^{(n)}(t) \geq \dots \geq \lambda_n^{(n)}(t)$, the spectral measure-valued process of $X_t^{(n)}$ is defined as

$$(1.4) \quad \mu_t^{(n)}(dx) = \frac{1}{n} \sum_{m=1}^n \delta_{\lambda_m^{(n)}(t)}(dx), t \geq 0.$$

Theorem 1. *Given a free Lévy process $\{Z_t : t \geq 0\}$, there exists a classical Lévy process $\{X_t : t \geq 0\}$ and an ensemble of Hermitian Lévy processes $\{X_t^{(n)} : t \geq 0\}_{n \geq 1}$, with $X_t^{(n)}$ an $n \times n$ matrix and $X_1^{(1)} \stackrel{\mathcal{L}}{=} X_1$, such that the spectral measure-valued processes $\{\mu_t^{(n)} : t \geq 0\}_{n \geq 1}$ of $\{X_t^{(n)} : t \geq 0\}_{n \geq 1}$ converge weakly in probability in the space $\mathcal{D}(\mathbb{R}_+, \Pr(\mathbb{R}))$ to the law of $\{Z_t : t \geq 0\}$. And the law of Z_1 corresponds to the image of the law of X_1 under the Bercovici-Pata bijection.*

As a particular case, we obtain a dynamical version of the Marchenko–Pastur theorem, where the asymptotic noncommutative process is the free Poisson process.

Remark 1. *The Hermitian Lévy processes $\{X_t^{(n)} : t \geq 0\}_{n \geq 1}$ in the above theorem satisfy the following properties:*

- (1) *For each $n \geq 1$ and $t > 0$, the matrix distribution of $X_t^{(n)}$ is absolutely continuous with respect to Lebesgue measure on \mathbb{R}^{2n} .*
- (2) *For each $n \geq 1$ and $t > 0$, the spectrum of $X_t^{(n)}$ is simple.*
- (3) *For each $t > 0$, $\{X_t^{(n)}\}_{n \geq 1}$ is a LUE; that is, for each $n \geq 1$, the matrix distribution of $X_t^{(n)}$ is infinitely divisible and invariant under unitary conjugation.*
- (4) *For each $n \geq 1$, the non-zero jumps of $X_t^{(n)}$, $\Delta X_t^{(n)} = X_t^{(n)} - X_{t-}^{(n)}$ are of rank one.*

Our random matrix models $\{X_1^{(n)}\}_{n \geq 1}$ are slightly different from those in [5], [12].

The strategy to prove Theorem 1 and the needed principal results are as follows. Section 2 contains the background on Hermitian Lévy processes, LUEs and free Lévy processes. A key result for the remainder of the paper is Lemma 1, which gives an estimate for the p -moment of the repulsion force between the eigenvalues of an $n \times n$ Hermitian Lévy process with an absolutely continuous distribution invariant under unitary conjugation. Section 3 introduces the ensembles of Hermitian Lévy processes $\{X_t^{(n)} : t \geq 0\}_{n \geq 1}$ of Theorem 1 via the appropriate characteristic triplets associated to the free Lévy process $\{Z_t : t \geq 0\}$ and with the properties (1)–(4) in Remark 1. Section 4 deals with the dynamics of the semimartingales of the eigenvalues of $X_t^{(n)}$, for which we use a recent Itô formula due to [24] and helpful asymptotics for the associated local martingales are also proved. Section 5 presents the proof of the tightness of the spectral measure-valued processes $\{\mu_t^{(n)} : t \geq 0\}_{n \geq 1}$ in the space $\mathcal{D}(\mathbb{R}_+, \Pr(\mathbb{R}))$, which, as expected, is more involved than the Brownian case due to the jumps. The key facts are that for each $n \geq 1$, all the jumps of the Hermitian Lévy process $\{X_t^{(n)} : t \geq 0\}$ are of rank one, used as well are useful estimates from Cabanal–Duvillard [12]. Finally, Theorem 2 in Section 6 identifies the Burger’s measure-valued equation satisfied by the limiting family of laws $\{\mu_t : t \geq 0\}$ of the spectral measure valued processes $\{\mu_t^{(n)} : t \geq 0\}_{n \geq 1}$ in $\mathcal{D}(\mathbb{R}_+, \Pr(\mathbb{R}))$, where $\{\mu_t : t \geq 0\}$ is the law of the free Lévy process $\{Z_t : t \geq 0\}$, employing a result of Bercovici and Voiculescu [7] for free infinitely divisible measures with unbounded support.

2. PRELIMINARIES ON HERMITIAN AND FREE LÉVY PROCESSES

2.1. Unitary invariant Hermitian Lévy processes. In this section we consider a class of Hermitian Lévy processes whose distributions are invariant under unitary conjugation.

Let $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ denote the linear space of $n \times n$ matrices with complex entries with scalar product $\langle A, B \rangle = \text{tr}(B^*A)$ and the Frobenius norm $\|A\| = [\text{tr}(A^*A)]^{1/2}$ where tr denotes the (non-normalized) trace. The set of Hermitian matrices in \mathbb{M}_n is denoted by \mathbb{H}_n , $\mathbb{H}_n^0 = \mathbb{H}_n \setminus \{0\}$ and \mathbb{H}_n^1 is the set of rank one matrices in \mathbb{H}_n . Let \mathbb{S}_n denote the unit sphere in \mathbb{H}_n , let $\mathbb{S}(\mathbb{H}_n^1) = \mathbb{S}_n \cap \mathbb{H}_n^1$ and let $\overline{\mathbb{H}}_n^+$ denote the set of nonnegative definite Hermitian matrices.

A random matrix X in \mathbb{H}_n is infinitely divisible if for all $m \geq 1$ there exist independent identically distributed random matrices X_1, \dots, X_m in \mathbb{H}_n such that $X_1 + \dots + X_m$ and the X have the same matrix distribution. In this case, the matrix distribution of X is characterized by the Lévy–Khintchine representation of the Fourier transform $\mathbb{E}e^{\text{itr}(\Theta X)} = \exp(\varphi(\Theta))$ with Laplace exponent

$$(2.5) \quad \varphi(\Theta) = \text{itr}(\Theta \Psi_n) - \frac{1}{2} \text{tr}(\Theta \mathcal{A}_n \Theta) + \int_{\mathbb{H}_n} \left(e^{\text{itr}(\Theta \xi)} - 1 - i \frac{\text{tr}(\Theta \xi)}{1 + \|\xi\|^2} \right) \nu_n(d\xi), \quad \Theta \in \mathbb{H}_n,$$

where $\mathcal{A}_n : \mathbb{H}_n \rightarrow \mathbb{H}_n$ is a linear operator which is positive (*i.e.* $\text{tr}(\Phi \mathcal{A}_n \Phi) \geq 0$ for $\Phi \in \mathbb{H}_n$) and symmetric (*i.e.* $\text{tr}(\Theta_2 \mathcal{A}_n \Theta_1) = \text{tr}(\Theta_1 \mathcal{A}_n \Theta_2)$ for $\Theta_1, \Theta_2 \in \mathbb{H}_n$), ν_n is a measure on \mathbb{H}_n (the Lévy measure) satisfying $\nu_n(\{0\}) = 0$ and $\int_{\mathbb{H}_n} (\|\xi\|^2 \wedge 1) \nu_n(d\xi) < \infty$, and $\Psi_n \in \mathbb{H}_n$. The triplet $(\mathcal{A}_n, \Psi_n, \nu_n)$ is unique.

The following is straightforward.

Proposition 1. Fix $n \geq 1$ and let X_n be an infinitely divisible $n \times n$ Hermitian random matrix with Lévy–Khintchine representation (2.5) with Lévy triplet $(\mathcal{A}_n, \Psi_n, \nu_n)$, where

- a) $\Psi_n = \gamma I_n, \gamma \in \mathbb{R}$,
- b) $\mathcal{A}_n \Theta = \frac{\sigma_n^2}{n} \Theta, \Theta \in \mathbb{H}_n, \sigma_n^2 \geq 0$, and
- c) $\nu_n(U E U^*) = \nu_n(E)$ for each unitary $n \times n$ nonrandom matrix U and $E \in \mathcal{B}(\mathbb{H}_n^0)$.

Then the distribution of X_n is invariant under unitary conjugation.

Definition 1. An $n \times n$ matrix-valued process $\{X(t) : t \geq 0\}$ is a Hermitian Lévy process if for each $t > 0$, $X(t) \in \mathbb{H}_n$ and

- i) $X(0) = 0$ with probability one,
- ii) X has independent increments: $\forall 0 \leq t_1 < \dots < t_m, m \geq 1, X(t_m) - X(t_{m-1}), \dots, X(t_2) - X(t_1)$ are independent random matrices,
- iii) X has stationary increments: $\forall 0 \leq s < t, X(t) - X(s)$ and $X(t-s)$ have the same matrix distribution, and
- iv) for any $s \geq 0$, the increment $X(t+s) - X(s) \rightarrow \mathbf{0}_n$ in distribution as $t \rightarrow 0$, where $\mathbf{0}_n$ is the $n \times n$ zero matrix.

A key feature of an $n \times n$ Hermitian Lévy process $X(t)$ with triplet given by Proposition 1 is that for each $t > 0$, the distribution of $X(t)$ is invariant under unitary conjugation. Furthermore, the nonzero jumps $\Delta X(t) = X(t) - X(t-)$ are random matrices of rank one.

Given any infinitely divisible $n \times n$ Hermitian random matrix X , there is a Hermitian Lévy process $\{X(t) : t \geq 0\}$ such that X and $X(1)$ have the same distribution, and vice versa. In fact, $X(t)$ has the Fourier transform $\mathbb{E}[e^{\text{itr}(\Theta X(t))}] = \exp(t\varphi(\Theta))$, where φ is the above Laplace exponent.

Throughout this paper we will assume that $X(t)$ has an absolutely continuous distribution for each $t > 0$. In order for this condition to hold, we will ask that X have a Gaussian component ($\sigma^2 \neq 0$) or that it satisfies condition **D** in [24]. Under this assumption, for each $t > 0$, $X(t)$ has a simple spectrum ([24]). Moreover, we obtain a very useful estimate for the p -moment of the repulsion force between the eigenvalues as follows.

Lemma 1. *Let $p \in [1, 2)$ and $T > 0$. Assume that $\{X(t) : t \geq 0\}$ is an $n \times n$ unitary invariant Hermitian Lévy process with absolutely continuous distribution. Let $(\lambda_1(t), \dots, \lambda_n(t))$ be the vector of eigenvalues of $X(t)$ where $\lambda_1(t) > \lambda_2(t) > \dots > \lambda_n(t)$ for each $t \geq 0$. Then there exists a constant $K_p > 0$ such that*

$$(2.6) \quad \mathbb{E} \left[\frac{1}{|(\lambda_r - \lambda_l)(s-)|^p} \right] < K_p,$$

for all $s \in [0, T]$, $r \neq l$. (K_p does not depend on either $s > 0$ or $n > 1$).

Proof. For $s \in [0, T]$, let f_s denote the density of $X(s)$. Then using the fact that $X(s)$ is invariant under unitary conjugation, we have from Lemma 4.1.6. in [17] that

$$\begin{aligned} \mathbb{E} \left(\frac{1}{|\lambda_r(s) - \lambda_l(s)|^p} \right) &= C_n \int_{\mathbb{R}^n} f_s(\lambda_1, \dots, \lambda_n) \prod_{i < j} \frac{(\lambda_i - \lambda_j)^2}{|\lambda_r - \lambda_l|^p} \prod_{i=1}^n d\lambda_i \\ &= C_n \int_{|\lambda_r - \lambda_l| > 1} f_s(\lambda_1, \dots, \lambda_n) \prod_{i < j} \frac{(\lambda_i - \lambda_j)^2}{|\lambda_r - \lambda_l|^p} \prod_{i=1}^n d\lambda_i + C_n \int_{|\lambda_r - \lambda_l| \leq 1} f_s(\lambda_1, \dots, \lambda_n) \prod_{i < j} \frac{(\lambda_i - \lambda_j)^2}{|\lambda_r - \lambda_l|^p} \prod_{i=1}^n d\lambda_i \\ &\leq C_n \int_{|\lambda_r - \lambda_l| > 1} f_s(\lambda_1, \dots, \lambda_n) \prod_{i < j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^n d\lambda_i + C_n \int_{|\lambda_r - \lambda_l| \leq 1} f_s(\lambda_1, \dots, \lambda_n) \prod_{i < j} \frac{(\lambda_i - \lambda_j)^2}{|\lambda_r - \lambda_l|^p} \prod_{i=1}^n d\lambda_i \\ &\leq C_n \int_{\mathbb{R}^n} f_s(\lambda_1, \dots, \lambda_n) \prod_{i < j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^n d\lambda_i + C_n \int_{|\lambda_r - \lambda_l| \leq 1} f_s(\lambda_1, \dots, \lambda_n) \prod_{i < j} \frac{(\lambda_i - \lambda_j)^2}{|\lambda_r - \lambda_l|^p} \prod_{i=1}^n d\lambda_i \\ &\leq 1 + \int_{|\lambda_r - \lambda_l| \leq 1} \frac{1}{|\lambda_r - \lambda_l|^p} d\lambda_r d\lambda_l. \end{aligned}$$

Now, by a change of variables to polar coordinates, we have

$$\begin{aligned} \int_{|\lambda_r - \lambda_l| \leq 1} \frac{1}{|\lambda_r - \lambda_l|^p} d\lambda_r d\lambda_l &\leq \int_0^{2\pi} \int_{r < |\cos \theta - \sin \theta|^{-1}} r^{1-p} \frac{1}{|\cos \theta - \sin \theta|^p} dr d\theta \\ &\leq \frac{1}{2-p} \int_0^{2\pi} \frac{1}{|\cos \theta - \sin \theta|^2} d\theta dr = \frac{1}{2-p} \frac{1}{2} \left[\tan \left(\frac{7\pi}{4} \right) - \tan \left(\frac{\pi}{4} \right) \right] := K_p < \infty. \end{aligned}$$

□

The following dynamics for the eigenvalues of a class of Hermitian Lévy processes was proved in [24].

Proposition 2. *Let $\{X(t) : t \geq 0\}$ be an $n \times n$ Hermitian Lévy process with absolutely continuous distribution invariant under unitary conjugation, and with triplet $(\sigma^2 \mathbf{I}_n \otimes \mathbf{I}_n, \gamma \mathbf{I}_n, \nu)$. Let $(\lambda_1(t), \dots, \lambda_n(t))$ be the vector of*

eigenvalues of $X(t)$ where $\lambda_1(t) > \lambda_2(t) > \dots > \lambda_n(t)$ for each $t \geq 0$. For each $m = 1, \dots, n$, the eigenvalue λ_m is a semimartingale and

$$(2.7) \quad \lambda_m(X_t) = \lambda_m(X_0) + \gamma \sum_{i=1}^n \int_0^t (D\lambda_m(X_{s-}))_{ii} ds + \sigma^2 \int_0^t \sum_{j \neq m} \frac{1}{\lambda_m(X_{s-}) - \lambda_j(X_{s-})} ds + M_t^m \\ + \int_{(0,t] \times \mathbb{H}_n^0} [\lambda_m(X_{s-} + y) - \lambda_m(X_{s-}) - \text{tr}(D\lambda_m(X_{s-})y)1_{\{\|y\| \leq 1\}}] \nu(dy) ds,$$

with

$$M_t^m = \sigma \sum_{r=1}^n \sum_{l=1}^n \int_0^t (D\lambda_m(X_{s-}))_{rl} dB_s^{rl} + \int_{(0,t] \times \mathbb{H}_n^0} (\lambda_m(X_{s-} + y) - \lambda_m(X_{s-})) \tilde{J}_X(ds, dy),$$

where $J_X(\cdot, \cdot)$ is the Poisson random measure of the jumps of X on $[0, \infty) \times \mathbb{H}_n^0$ with intensity measure $\text{Leb} \otimes \nu$, independent of a family of independent one dimensional standard Brownian motions B_s^{ij} , $i, j = 1, \dots, n$ and the compensated measure is given by

$$\tilde{J}_X(dt, dy) = J_X(dt, dy) - dt\nu(dy);$$

and for each $s \geq 0$, $D\lambda_m(X_s)$ is the matrix of derivatives of $\lambda_m(X_s)$ with respect to the entries of X_s , given by

$$(2.8) \quad (D\lambda_m(X_s))_{ij} = 2\bar{u}_{im}(s)u_{jm}(s)1_{\{i < j\}} + |u_{im}|^2 1_{\{i=j\}},$$

where $u_{ij}(s)$ $i, j = 1, 2, \dots, n$ are the entries of a unitary random matrix.

Remark 2. If we take

$$\widetilde{M}_t^m := \sum_{r=1}^n \sum_{l=1}^n \int_0^t (D\lambda_m(X_{s-}))_{rl} dB_s^{rl},$$

it is clear that for $m, m' = 1, \dots, n$ its covariation process is given by

$$\langle \widetilde{M}^m, \widetilde{M}^{m'} \rangle_t = t\delta_{mm'} \quad t > 0.$$

Therefore, by Lévy's Theorem, we can write, for $m = 1, \dots, n$, the martingale term M^m as

$$M_t^m = \sigma W_t^m + \int_{(0,t] \times \mathbb{H}_n^0} (\lambda_m(X_{s-} + y) - \lambda_m(X_{s-})) \tilde{J}_X(ds, dy), \quad \text{for } t > 0,$$

where W^1, \dots, W^n are independent one dimensional standard Brownian motions.

2.2. Free Lévy processes affiliated with W^* -probability spaces. Next we recall some facts on free Lévy processes acting on a W^* -probability space. For additional information on this subject, see [1], [4], [7]. A W^* -probability space is a pair (\mathcal{G}, τ) where \mathcal{G} is a von Neumann algebra acting on a Hilbert space H and τ is a normal faithful trace on \mathcal{G} . In the sequel, (\mathcal{G}, τ) will denote a W^* -probability space.

The problem of considering a probability measure on \mathbb{R} with unbounded support as the spectral distribution of some selfadjoint operator leads to considering unbounded operators. A linear operator a in H is a (generally unbounded) linear operator $a : D \rightarrow H$ defined on a subspace $D \subset H$. An unbounded operator a in H is not an element of \mathcal{G} . However, a selfadjoint linear operator a in H is affiliated with \mathcal{G} if and only if $f(a) \in \mathcal{G}$ for any bounded Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$. Here $f(a)$ is defined in the sense of spectral theory (the functional

calculus). That is, for any selfadjoint operator a affiliated with \mathcal{G} , there exists a unique probability measure μ_a on \mathbb{R} , concentrated on the spectrum of a , such that

$$\tau(f(a)) = \int_{\mathbb{R}} f(s) \mu_a(ds),$$

for every bounded Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$. The measure μ_a is called the (spectral) distribution of a and is denoted by $\mu_a = \mathcal{L}\{a\}$. Unless a is bounded, the spectrum of a is an unbounded subset of \mathbb{R} and, in general, μ_a is not compactly supported.

Definition 2. Let a_1, a_2, \dots, a_r be selfadjoint operators affiliated with a W^* -probability space (\mathcal{G}, τ) . It is said that a_1, a_2, \dots, a_r are freely independent with respect to τ if for any bounded Borel functions $f_1, f_2, \dots, f_r : \mathbb{R} \rightarrow \mathbb{R}$, the bounded linear operators $f_1(a_1), f_2(a_2), \dots, f_r(a_r)$ in \mathcal{G} are freely independent with respect to τ . That is,

$$(2.9) \quad \tau \left\{ [f_{i_1}(a_{i_1}) - \tau(f_{i_1}(a_{i_1}))] [f_{i_2}(a_{i_2}) - \tau(f_{i_2}(a_{i_2}))] \cdots [f_{i_m}(a_{i_m}) - \tau(f_{i_m}(a_{i_m}))] \right\} = 0$$

for any positive integer m and any i_1, i_2, \dots, i_m in $\{1, 2, \dots, r\}$ with $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{m-1} \neq i_m$.

A stochastic process affiliated with a W^* -probability space (\mathcal{G}, τ) is a family $\{Z_t : t \geq 0\}$ of selfadjoint operators affiliated with \mathcal{G} . Let us denote by $\mu_t = \mathcal{L}\{Z_t\}$ the (spectral) distribution of Z_t for each $t \geq 0$. The family $\{\mu_t : t \geq 0\}$ of probability measures on \mathbb{R} is called the family of spectral distributions of the process $\{Z_t : t \geq 0\}$. Moreover, for any $s \geq 0, t \geq 0$ such that $s \leq t$, the increment $Z_t - Z_s$ is again a selfadjoint operator affiliated with \mathcal{G} and we denote its distribution by $\mu_{s,t} = \mathcal{L}\{Z_t - Z_s\}$.

Definition 3. A free Lévy process is a stochastic process $\{Z_t : t \geq 0\}$ affiliated with the W^* -probability space (\mathcal{G}, τ) such that:

- i) $Z_0 = 0$.
- ii) For any $m \geq 1$ and $0 \leq t_1 < \dots < t_m$, the increments

$$Z_{t_m} - Z_{t_{m-1}}, \dots, Z_{t_2} - Z_{t_1}$$

are freely independent random variables.

- iii) For any $s \geq 0, t \geq 0$ the spectral distribution of $Z_{t+s} - Z_s$ does not depend on s .

iv) For any $s \geq 0$, the increment $Z_{t+s} - Z_s \rightarrow 0$ in distribution as $t \rightarrow 0$, that is, the spectral distributions $\mathcal{L}\{Z_{t+s} - Z_s\}$ converge weakly to δ_0 as $t \rightarrow 0$.

It is well known that the law $\nu = \mathcal{L}(Z_1)$ of a free Lévy process $\{Z_t : t \geq 0\}$ is free infinitely divisible. Moreover, it has the Lévy–Khinchine representation $\phi_{Z_t}(z) = t\phi_\nu(z)$ in terms of the Voiculescu transform

$$(2.10) \quad \phi_\nu(z) = \eta + \int_{\mathbb{R}} \frac{1+tz}{z-t} \rho(dt), \quad (z \in \mathbb{C}^+),$$

with generating pair (η, ρ) , where $\eta \in \mathbb{R}$ and ρ is a finite measure on \mathbb{R} , see [4], [7], [30].

Finally, the Bercovici–Pata bijection [6] Λ between the set of classical one-dimensional infinitely divisible distributions $ID(*)$ and the set of free infinitely divisible distributions $ID(\boxplus)$, is such that for each $\mu \in ID(*)$

with Lévy triplet (σ^2, γ, ν) , $\Lambda(\mu) \in ID(\boxplus)$ has generating pair

$$(2.11) \quad \rho(dx) = \sigma^2 \delta_0(dx) + \frac{x^2}{1+x^2} \nu(dx),$$

$$(2.12) \quad \eta = \gamma - \int_{\mathbb{R}} x \left(1_{[-1,1]}(x) - \frac{1}{1+x^2} \right) \nu(dx),$$

see [3], [4].

3. THE APPROXIMATING HERMITIAN LÉVY PROCESSES

In this section we introduce the ensemble of Hermitian measure valued processes considered in this paper. Let (η, ρ) be the generating pair of a free Lévy process $\{Z_t : t \geq 0\}$ and let $\sigma^2 = \rho(\{0\})$. For each $n \geq 1$ we construct an $n \times n$ Hermitian Lévy process $X^{(n)} = \{X_t^{(n)} : t \geq 0\}$ with generating triplet $\left(\frac{\sigma_n^2}{n} \mathbf{I}_n \otimes \mathbf{I}_n, \gamma \mathbf{I}_n, \nu_n\right)$ given by

a)

$$(3.13) \quad \sigma_n^2 = \sigma^2 + \frac{n-1}{n^2},$$

b)

$$(3.14) \quad \gamma = \eta + \int_{|r| \leq 1} r \rho(dr) - \int_{|r| > 1} \frac{1}{r} \rho(dr).$$

c) The Lévy measure ν_n has the polar decomposition (π_n, ρ_n^α) ,

$$(3.15) \quad \nu_n(E) = \int_{\mathbb{S}(\mathbb{H}_n^1)} \int_0^\infty 1_E(r\xi) \rho_n^\alpha(dr) \pi_n(d\xi), \quad E \in \mathcal{B}(\mathbb{H}_n \setminus \{0\}),$$

where:

i) The spherical measure π_n is concentrated on $\mathbb{S}(\mathbb{H}_n^1)$ and satisfies

$$(3.16) \quad \int_{\mathbb{S}(\mathbb{H}_n^1)} 1_B(\xi) \pi_n(d\xi) = n \int_{\mathbb{S}(\mathbb{C}_n)} 1_{\phi_n^{-1}(B)}(u) \pi(du), \quad B \in \mathcal{B}(\mathbb{S}(\mathbb{H}_n^1))$$

where ϕ_n denotes the transformation $u \rightarrow uu^*$ and π is the Haar distribution of a random vector in $\mathbb{S}(\mathbb{C}_n)$, the unit sphere of \mathbb{C}_n .

ii) The radial component ρ_n^α is a measure defined on $(0, \infty)$ for each $n \geq 1$ and $\alpha \in (0, 1/2)$ by

$$\rho_n^\alpha(dr) = \frac{1+r^2}{r^2} [\rho(-dr) + \rho(dr)] 1_{(0, n^{2\alpha}/(n^\alpha-1))}(r).$$

where the above expression is understood in the limiting sense when $n = 1$. Note that $\int_{-\infty}^\infty (1 \wedge r^2) \rho_n^\alpha(dr) < \infty$.

Remark 3. Note that in the case $n = 1$ we have that

$$\begin{aligned} \sigma_1^2 &= \sigma^2 = \rho(\{0\}) \\ \rho_1^\alpha &= \frac{1+r^2}{r^2} [\rho(-dr) + \rho(dr)] \quad \text{and,} \\ \gamma &= \eta + \int_{|r| \leq 1} r \rho(dr) - \int_{|r| > 1} \frac{1}{r} \rho(dr), \end{aligned}$$

which is the generating triplet, corresponding to the pair (η, ρ) , of the free Lévy process $\{Z_t : t \geq 0\}$ under the Bercovici-Pata bijection.

Remark 4. i) $X_t^{(n)}$ has an absolutely continuous distribution with respect to Lebesgue measure on \mathbb{R}^{2n} , since σ_n^2 in (3.13) is non-zero.

ii) The absolute continuity implies that the spectrum of $X_t^{(n)}$ is simple for each $t > 0$, $n > 1$, see [24].

iii) $X_t^{(n)}$ has a unitary invariant distribution, for each $t > 0$, $n \geq 1$, since the assumptions of Proposition 1 are satisfied. This follows since the spherical measure π_n is a multiple of the Haar distribution and ρ_n^α does not depend on $\xi \in \mathbb{S}(\mathbb{H}_n^1)$.

iv) The sequence of Hermitian matrices $\{X^{(n)}\}_{n \geq 1}$ is a LUE.

v) The non-zero jumps of $X^{(n)}$ are of rank one, i.e., $\Delta X^{(n)}(t) = X^{(n)}(t) - X^{(n)}(t-) \in \mathbb{H}_n^1$. This follows since the spherical measure π_n is concentrated on $\mathbb{S}(\mathbb{H}_n^1)$.

Example. When Z is the free Poisson process, $\eta = -\lambda$, $\rho = \lambda \delta_1$, $\lambda > 0$. Then, for each $n \geq 1$

$$X_t^{(n)} = \sum_{j=1}^{N_t} u_j u_j^* + \frac{n-1}{n^2} B_t I_n$$

where $\{u_j\}_{j \geq 1}$ is a sequence of independent uniformly distributed random vectors in \mathbb{C}^n , $N = \{N_t\}_{t \geq 0}$ is a Poisson process of parameter λ independent of $\{u_j\}_{j \geq 1}$ and $B = \{B_t\}_{t \geq 0}$ is a one-dimensional standard Brownian motion independent of N and $\{u_j\}_{j \geq 1}$. Then Theorem 1 gives a dynamical version of the Marchenko–Pastur theorem, in which the asymptotic noncommutative process is the free Poisson process. We point out that the dynamical version in [13] gives as an asymptotic process the dilation of the free Poisson distribution, which is not a free Lévy process.

4. THE DYNAMICS OF THE EIGENVALUES AND THE MEASURE VALUED PROCESSES

In this section we establish several results that are needed later on for the sequence of semimartingales corresponding to the eigenvalues and the spectral measure valued processes of the ensemble of the Hermitian Lévy process $\{X_t^{(n)} : t \geq 0\}_{n \geq 1}$ defined in the last section. Throughout this section and in the rest of the paper we will use the following notation

$$\langle \mu, f \rangle := \int_{\mathbb{R}} f(x) \mu(dx),$$

for any bounded measurable function f and $\mu \in \text{Pr}(\mathbb{R})$.

For each $t \geq 0$ let $\lambda_1^{(n)}(t) > \dots > \lambda_n^{(n)}(t)$ denote the eigenvalues of $X^{(n)}$. Then, by Proposition 2,

$$\begin{aligned} \lambda_m^{(n)}(X_t^{(n)}) &= \lambda_m(X_0^{(n)}) + \gamma \sum_{i=1}^n \int_0^t (D\lambda_m^{(n)}(X_{s-}^{(n)}))_{ii} ds + \frac{\sigma_n^2}{n} \int_0^t \sum_{j \neq m} \frac{1}{(\lambda_m^{(n)} - \lambda_j^{(n)})(s-)} ds + M_t^{n,m} \\ (4.17) \quad &+ \int_0^t \int_{\mathbb{S}(\mathbb{H}_n^1)} \int_{\mathbb{R}} \left[\lambda_m^{(n)}(X_{s-}^{(n)} + ry) - \lambda_m^{(n)}(X_{s-}^{(n)}) - \text{tr}(D\lambda_m^{(n)}(X_{s-}^{(n)})ry) \right] 1_{\{|r| \leq 1\}} \rho_n^\alpha(dr) \pi_n(dy) ds, \end{aligned}$$

where the process $(M_t^{n,m})_{t \geq 0}$ is a martingale.

Let $\mu_t^{(n)}$ be the corresponding spectral measure-valued process (1.4) of $X^{(n)}(t)$. Let $f \in \mathcal{C}_b^2(\mathbb{R})$ (where $\mathcal{C}_b^2(\mathbb{R})$ denotes the set of twice differentiable functions with bounded derivatives). Then from (4.17) and an application of

Itô's formula we obtain that

$$\begin{aligned}
\langle \mu_t^{(n)}, f \rangle &= \langle \mu_0^{(n)}, f \rangle + \frac{1}{n} \sum_{m=1}^n \int_0^t f'(\lambda_m^{(n)}(X_{s-}^{(n)})) dM_s^{n,m} + \frac{\sigma_n^2}{n^2} \sum_{m=1}^n \int_0^t f'(\lambda_m^{(n)}(X_{s-}^{(n)})) \sum_{j \neq m} \frac{1}{(\lambda_m^{(n)} - \lambda_j^{(n)})(s-)} ds \\
&+ M_t^{n,f} + \frac{\gamma}{n} \sum_{m=1}^n \sum_{i=1}^n \int_0^t f'(\lambda_m^{(n)}(X_{s-}^{(n)})) (D\lambda_m^{(n)}(X_{s-}^{(n)}))_{ii} ds + \frac{1}{2n} \sum_{m=1}^n \int_0^t f''(\lambda_m^{(n)}(X_{s-}^{(n)})) d\langle M^{n,m}, M^{n,m} \rangle_s^c \\
&+ \frac{1}{n} \sum_{m=1}^n \int_0^t f'(\lambda_m^{(n)}(X_{s-}^{(n)})) \int_{\mathbb{S}(\mathbb{H}_n^1)} \int_{\mathbb{R}} [\lambda_m^{(n)}(X_{s-}^{(n)} + ry) \\
&\quad - \lambda_m^{(n)}(X_{s-}^{(n)}) - \text{tr}(D\lambda_m^{(n)}(X_{s-}^{(n)})ry) 1_{\{|r| \leq 1\}}] \rho_n^\alpha(dr) \pi_n(dy) ds \\
(4.18) \quad &+ \frac{1}{n} \sum_{m=1}^n \int_0^t \int_{\mathbb{S}(\mathbb{H}_n^1)} \int_{\mathbb{R}} \left[f(\lambda_m^{(n)}(X_{s-}^{(n)} + ry)) - f(\lambda_m^{(n)}(X_{s-}^{(n)})) - f'(\lambda_m^{(n)}(X_{s-}^{(n)})) \Delta \lambda_m^{(n)}(X_{s-}^{(n)}) \right] \rho_n^\alpha(dr) \pi_n(dy) ds,
\end{aligned}$$

where $(M_t^{n,f})_{t \geq 0}$ is a local martingale.

Following Remark 2 it is easy to check that

$$\frac{1}{2n} \sum_{m=1}^n \int_0^t f''(\lambda_m^{(n)}(X_{s-}^{(n)})) d\langle M^{n,m}, M^{n,m} \rangle_s^c = \frac{\sigma_n^2}{2n^2} \sum_{m=1}^n \int_0^t f''(\lambda_m^{(n)}(X_{s-}^{(n)})) ds,$$

and therefore

$$\begin{aligned}
&\frac{\sigma_n^2}{n^2} \sum_{m=1}^n \int_0^t f'(\lambda_m^{(n)}(X_{s-}^{(n)})) \sum_{j \neq m} \frac{1}{(\lambda_m^{(n)} - \lambda_j^{(n)})(s-)} ds + \frac{1}{2n} \sum_{m=1}^n \int_0^t f''(\lambda_m^{(n)}(X_{s-}^{(n)})) d\langle M^{n,m}, M^{n,m} \rangle_s^c \\
&= \frac{\sigma_n^2}{2} \int_0^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_s^n(dx) \mu_s^n(dy) ds.
\end{aligned}$$

Using (2.8) the drift term is expressed as

$$\begin{aligned}
\frac{\gamma}{n} \sum_{m=1}^n \sum_{i=1}^n \int_0^t f'(\lambda_m^{(n)}(X_{s-}^{(n)})) (D\lambda_m^{(n)}(X_{s-}^{(n)}))_{ii} ds &= \frac{\gamma}{n} \sum_{m=1}^n \sum_{i=1}^n \int_0^t f'(\lambda_m^{(n)}(X_{s-}^{(n)})) |\bar{u}_{im}^{(n)} u_{im}^{(n)}| ds \\
&= \frac{\gamma}{n} \sum_{m=1}^n \int_0^t f'(\lambda_m^{(n)}(X_{s-}^{(n)})) ds = \gamma \int_0^t \int_{\mathbb{R}} f'(x) \mu_s^{(n)}(dx) ds.
\end{aligned}$$

Now using (3.15) and (3.16) the last two terms in (4.18) can be written as

$$\begin{aligned}
&\sum_{m=1}^n \int_0^t \int_{\mathbb{S}(\mathbb{C}_n)} \int_{\mathbb{R}} \left[f(\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*)) - f(\lambda_m^{(n)}(X_{s-}^{(n)})) \right. \\
&\quad \left. - f'(\lambda_m^{(n)}(X_{s-}^{(n)})) \text{tr}(D\lambda_m^{(n)}(X_{s-}^{(n)}) r v v^*) 1_{\{|r| \leq 1\}} \right] \rho_n^\alpha(dr) \pi(dv) ds.
\end{aligned}$$

Thus (4.18) is expressed as

$$\begin{aligned}
 \langle \mu_t^{(n)}, f \rangle &= \langle \mu_0^n, f \rangle + \frac{\sigma_n^2}{2} \int_0^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_s^n(dx) \mu_s^n(dy) ds + \gamma \int_0^t \int_{\mathbb{R}} f'(x) \mu_s^n(dx) ds + M_t^{n,f} \\
 &+ \sum_{m=1}^n \int_0^t \int_{\mathbb{S}(\mathbb{C}_n)} \int_{\mathbb{R}} \left[f(\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*)) - f(\lambda_m^{(n)}(X_{s-}^{(n)})) \right. \\
 &\quad \left. - f'(\lambda_m^{(n)}(X_{s-}^{(n)})) \text{tr}(D\lambda_m^{(n)}(X_{s-}^{(n)}) r v v^*) 1_{\{|r| \leq 1\}} \right] \rho_n^\alpha(dr) \pi(dv) ds,
 \end{aligned}
 \tag{4.19}$$

where the martingale term $(M_t^{n,f})_{t \geq 0}$ is given by

$$\begin{aligned}
 M_t^{n,f} &= \frac{\sigma_n}{n^{3/2}} \sum_{m=1}^n \int_0^t f'(\lambda_m^{(n)}(X_{s-}^{(n)})) dW_s^m \\
 &+ \frac{1}{n} \sum_{m=1}^n \int_0^t \int_{\mathbb{S}(\mathbb{H}_n^1)} \int_{\mathbb{R}} f(\lambda_m^{(n)}(X_{s-}^{(n)})) \left[\lambda_m^{(n)}(X_{s-}^{(n)} + r y) - \lambda_m^{(n)}(X_{s-}^{(n)}) \right] \tilde{J}_X(ds, dr, dy),
 \end{aligned}
 \tag{4.20}$$

where $W^m, m = 1, \dots, n$ are independent one-dimensional Brownian motions.

Lemma 2. *For any bounded continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$, the martingale $(M_t^{n,f})_{t \geq 0}$ in (4.20) satisfies*

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} |M_t^{n,f}| = 0 \quad \text{in probability,}$$

for any $T > 0$.

Proof. We show the convergence of each term in (4.20). Let $\varepsilon > 0$. By Doob's inequality and (2.8)

$$\begin{aligned}
 \mathbb{P} \left(\sup_{0 \leq t \leq T} \left| \frac{\sigma_n}{n^{3/2}} \int_0^t \sum_{m=1}^n f'(\lambda_m^{(n)}(X_{s-}^{(n)})) dW_s^m \right| > \varepsilon \right) &\leq \frac{1}{\varepsilon^2} \frac{\sigma_n^2}{n^3} \mathbb{E} \left[\left(\int_0^T \sum_{m=1}^n f'(\lambda_m^{(n)}(X_{s-}^{(n)})) dW_s^m \right)^2 \right] \\
 &\leq \frac{1}{\varepsilon^2} \frac{\sigma_n^2}{n^3} \mathbb{E} \left[\int_0^T \sum_{m=1}^n (f'(\lambda_m^{(n)}(X_{s-}^{(n)})))^2 ds \right] \leq \frac{1}{\varepsilon^2} \frac{(2\sigma + 1)^2}{n^2} \|f'\|_\infty^2 T,
 \end{aligned}$$

which converges to 0 as $n \rightarrow \infty$.

Let us consider $n > 1$, then by an application of Doob's inequality in the second term:

$$\begin{aligned}
 \mathbb{P} \left(\sup_{0 \leq t \leq T} \left| \frac{1}{n} \sum_{m=1}^n \int_0^t \int_{\mathbb{S}(\mathbb{C}_n)} \int_{\mathbb{R}} f(\lambda_m^{(n)}(X_{s-}^{(n)})) \left[\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*) - \lambda_m^{(n)}(X_{s-}^{(n)}) \right] \tilde{J}_X(ds, dr, dv) \right| > \varepsilon \right) \\
 \leq \frac{1}{n^2} \frac{1}{\varepsilon^2} \mathbb{E} \left[\left(\sum_{m=1}^n \int_0^T \int_{\mathbb{S}(\mathbb{C}_n)} f(\lambda_m^{(n)}(X_{s-}^{(n)})) \left[\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*) - \lambda_m^{(n)}(X_{s-}^{(n)}) \right] \tilde{J}_X(ds, dr, dv) \right)^2 \right] \\
 \leq \frac{1}{n} \frac{1}{\varepsilon^2} \|f\|_\infty^2 \mathbb{E} \int_0^T \int_{\mathbb{S}(\mathbb{C}_n)} \int_{\mathbb{R}} \left(\sum_{m=1}^n \left[\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*) - \lambda_m^{(n)}(X_{s-}^{(n)}) \right] \right)^2 \rho_n^\alpha(dr) \pi(dv) ds,
 \end{aligned}$$

where we have used (3.16). Using Lemma III.5 in [12] we get

$$\begin{aligned}
&\leq \frac{1}{n} \frac{1}{\varepsilon^2} \|f\|_\infty^2 \int_0^T \int_{\mathbb{S}(\mathbb{C}_n)} \int_{\mathbb{R}} r^2 \rho_n^\alpha(dr) \pi(dv) ds \leq \|f\|_\infty^2 \frac{1}{n} \frac{1}{\varepsilon^2} T \int_{\mathbb{R}} r^2 \rho_n^\alpha(dr) \\
&\leq \|f\|_\infty^2 \frac{1}{n} \frac{1}{\varepsilon^2} T \left(\int_{|r| \leq 1} r^2 \rho_n^\alpha(dr) + \int_{1 \leq |r|} r^2 \rho_n^\alpha(dr) \right) \\
&\leq \|f\|_\infty^2 \frac{1}{n} \frac{1}{\varepsilon^2} T \left(\int_{|r| \leq 1} r^2 \rho_n^\alpha(dr) + \int_{1 \leq |r| \leq n^{2\alpha}/(n^\alpha-1)} (1+r^2) \rho(dr) \right) \leq \frac{n^{4\alpha}}{n(n^\alpha-1)^2} C(f, T)
\end{aligned}$$

for some constant $C(f, T) > 0$, and hence $\frac{n^{4\alpha}}{n(n^\alpha-1)^2} C(f, T) \rightarrow 0$ as $n \rightarrow \infty$. This concludes the proof. \square

5. TIGHTNESS

Let $\{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\}$ be the family of measure valued-processes (1.4) of the Hermitian Lévy process ensemble $(X^{(n)})_{n \geq 1}$ introduced in Section 4. In this section we prove that this family is tight in the space $\mathcal{D}(\mathbb{R}_+, \text{Pr}(\mathbb{R}))$.

The keys to the proof are the eigenvalue semimartingale estimates of Section 5, the fact that for each $n \geq 1$ all the jumps of the Hermitian Lévy process $\{X_t^{(n)} : t \geq 0\}$ are of rank one, and an estimate given in Lemma III.5 in [12], which allows a useful bound on $|\text{tr}(f(X_t^{(n)} + X_s^{(n)})) - \text{tr}(f(X_t^{(n)}))|$ when f is a Lipschitz function.

Proposition 3. *The family of measures $\{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\}$ is tight in the space $\mathcal{D}(\mathbb{R}_+, \text{Pr}(\mathbb{R}))$.*

Proof. In order to prove tightness of the family of laws $\{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\}$ in $\mathcal{D}(\mathbb{R}_+, \text{Pr}(\mathbb{R}))$, we use the Aldous–Rebolledo Criterion (see [2], [16], [26]) to prove that for each, $f \in \mathcal{C}_b^2(\mathbb{R})$ with bounded variation, the sequence of real processes $\{(\langle \mu_t^{(n)}, f \rangle)_{t \geq 0} : n \geq 1\}$ is tight. We split the proof of the tightness of the semimartingale $\langle \mu^{(n)}, f \rangle$ into two steps: the first is on the bounded variation part and the second on the martingale part.

For any $f \in \mathcal{C}_b^2(\mathbb{R})$ with finite total variation, we have from (4.19) that

$$\begin{aligned}
\langle \mu_t^{(n)}, f \rangle - \langle \mu_s^{(n)}, f \rangle &= \frac{\sigma_n^2}{2} \int_s^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_u^{(n)}(dx) \mu_u^{(n)}(dy) du + \gamma \int_s^t \int_{\mathbb{R}} f'(x) \mu_u^{(n)}(dx) du \\
&+ M_t^{n,f} - M_s^{n,f} + \sum_{m=1}^n \int_s^t \int_{\mathbb{S}(\mathbb{C}_n)} \int_{\mathbb{R}} \left[f(\lambda_m^{(n)}(X_{u-}^{(n)} + r v v^*)) - f(\lambda_m^{(n)}(X_{u-}^{(n)})) \right. \\
&\quad \left. - f'(\lambda_m^{(n)}(X_{u-}^{(n)})) \text{tr}(D \lambda_m^{(n)}(X_{u-}^{(n)}) r v v^*) 1_{\{|r| \leq 1\}} \right] \rho_n^\alpha(dr) \pi(dv) du.
\end{aligned}$$

Let us denote by $V^{n,f}$ the bounded variation part of the semimartingale $\langle \mu^{(n)}, f \rangle$. Then, for each $0 \leq s \leq t$,

$$\begin{aligned}
V_t^{n,f} - V_s^{n,f} &= \frac{\sigma_n^2}{2} \int_s^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_u^{(n)}(dx) \mu_u^{(n)}(dy) du + \gamma \int_s^t \int_{\mathbb{R}} f'(x) \mu_u^{(n)}(dx) du \\
&+ \sum_{m=1}^n \int_s^t \int_{\mathbb{S}(\mathbb{C}_n)} \int_{\mathbb{R}} \left[f(\lambda_m^{(n)}(X_{u-}^{(n)} + r v v^*)) - f(\lambda_m^{(n)}(X_{u-}^{(n)})) \right. \\
&\quad \left. - f'(\lambda_m^{(n)}(X_{u-}^{(n)})) \text{tr}(D \lambda_m^{(n)}(X_{u-}^{(n)}) r v v^*) 1_{\{|r| \leq 1\}} \right] \rho_n^\alpha(dr) \pi(dv) du.
\end{aligned} \tag{5.21}$$

Let $\delta > 0$ and $\theta \in [0, \delta]$. Let $T' > 0$ and let $(\tau_n)_{n \geq 1}$ be a sequence of stopping times such that $0 \leq \tau_n < T'$.

Next we estimate each term of (5.21). For the first term we have

$$(5.22) \quad \left| \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_u^{(n)}(dx) \mu_u^{(n)}(dy) du \right| = \left| \frac{1}{n^2} \sum_{m=1}^n \sum_{m'=1}^n \int_{\tau_n}^{\tau_n+\theta} \frac{f'(\lambda_m^{(n)}(X_{u-}^{(n)})) - f'(\lambda_{m'}^{(n)}(X_{u-}^{(n)}))}{\lambda_m^{(n)}(u) - \lambda_{m'}^{(n)}(u)} du \right|$$

$$\leq \frac{1}{n^2} \sum_{m=1}^n \sum_{m'=1}^n \int_{\tau_n}^{\tau_n+\theta} \frac{|f'(\lambda_m^{(n)}(X_{u-}^{(n)})) - f'(\lambda_{m'}^{(n)}(X_{u-}^{(n)}))|}{|\lambda_m^{(n)}(u) - \lambda_{m'}^{(n)}(u)|} du,$$

and by the mean value theorem there exists $\xi \in \mathbb{R}$ such that

$$|f'(\lambda_m^{(n)}(X_{u-}^{(n)})) - f'(\lambda_{m'}^{(n)}(X_{u-}^{(n)}))| \leq |f''^{(n)}(\xi)| |\lambda_m^{(n)}(u) - \lambda_{m'}^{(n)}(u)| \leq \|f''\|_{\infty} |\lambda_m^{(n)}(u) - \lambda_{m'}^{(n)}(u)|,$$

now from (5.22),

$$(5.23) \quad \left| \sigma_n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_u^{(n)}(dx) \mu_u^{(n)}(dy) du \right| \leq (1 + \sigma^2)^{1/2} \|f''\|_{\infty} \delta.$$

For the second term in (5.21),

$$(5.24) \quad \left| \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{R}} f'(x) \mu_s^n(dx) ds \right| = \left| \frac{1}{n} \sum_{m=1}^n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{R}} f'(\lambda_m^{(n)}(X_{u-}^{(n)})) du \right| \leq \|f'\|_{\infty} \delta.$$

For the jump part in (5.21) let us consider the associated term

$$(5.25) \quad \begin{aligned} & \sum_{m=1}^n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| \leq 1} \left[f(\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*)) - f(\lambda_m^{(n)}(X_{s-}^{(n)})) \right. \\ & \quad \left. - f'(\lambda_m^{(n)}(X_{s-}^{(n)})) \text{tr}(D\lambda_m^{(n)}(X_{s-}^{(n)}) r v v^*) \right] \rho_n^{\alpha}(dr) \pi(dv) ds \\ & = \sum_{m=1}^n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| \leq 1} \left[f(\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*)) - f(\lambda_m^{(n)}(X_{s-}^{(n)})) - f'(\lambda_m^{(n)}(X_{u-}^{(n)})) \right. \\ & \quad \left. \times (\lambda_m^{(n)}(X_{s-} + r v v^*) - \lambda_m^{(n)}(X_{s-})) \right] \rho_n^{\alpha}(dr) \pi(dv) ds \\ & + \sum_{m=1}^n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| \leq 1} \left[f'(\lambda_m^{(n)}(X_{u-}^{(n)})) \Delta \lambda_m^{(n)}(X_{s-}^{(n)}) \right. \\ & \quad \left. - f'(\lambda_m^{(n)}(X_{s-}^{(n)})) \text{tr}(D\lambda_m^{(n)}(X_{s-}^{(n)}) r v v^*) \right] \rho_n^{\alpha}(dr) \pi(dv) ds, \end{aligned}$$

Now we estimate both terms of (5.25). For the first one we have by Taylor's theorem that there exists $\xi \in \mathbb{R}$ such that

$$\begin{aligned} & \left| f(\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*)) - f(\lambda_m^{(n)}(X_{s-}^{(n)})) - f'(\lambda_m^{(n)}(X_{s-}^{(n)})) (\lambda_m^{(n)}(X_{s-} + r v v^*) - \lambda_m^{(n)}(X_{s-})) \right| \\ & = \frac{1}{2} |f''(\xi)| (\lambda_m^{(n)}(X_{s-} + r v v^*) - \lambda_m^{(n)}(X_{s-}))^2 \\ & \leq \frac{1}{2} \|f''\| (\lambda_m^{(n)}(X_{s-} + r v v^*) - \lambda_m^{(n)}(X_{s-}))^2, \end{aligned}$$

and therefore

$$\begin{aligned}
& \mathbb{E} \left[\left| \sum_{m=1}^n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| \leq 1} \left[f(\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*)) - f(\lambda_m^{(n)}(X_{s-}^{(n)})) - f'(\lambda_m^{(n)}(X_{s-}^{(n)})) \right. \right. \right. \\
& \quad \left. \left. \left. \times \left(\lambda_m^{(n)}(X_{s-} + r v v^*) - \lambda_m^{(n)}(X_{s-}) \right) \right] \rho_n^\alpha(dr) \pi(dv) ds \right| \right] \\
& \leq \frac{1}{2} \|f''\|_\infty \mathbb{E} \left[\sum_{m=1}^n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| \leq 1} \left(\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*) - \lambda_m^{(n)}(X_{s-}^{(n)}) \right)^2 \rho_n^\alpha(dr) \pi(dv) ds \right] \\
& \leq \frac{1}{2} \|f''\|_\infty \mathbb{E} \left[\sum_{i,j=1}^n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| \leq 1} \left(X_{s-}^{(n)} + r v v^* - X_{s-}^{(n)} \right)_{ij}^2 \rho_n^\alpha(dr) \pi(dv) ds \right] \\
& \leq \frac{1}{2} \|f''\|_\infty \mathbb{E} \left[\sum_{i,j=1}^n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| \leq 1} (r v v^*)_{ij}^2 \rho_n^\alpha(dr) \pi(dv) du \right] \\
& \leq \frac{1}{2} \|f''\|_\infty \mathbb{E} \left[\sum_{i,j=1}^n \int_{\tau_n}^{\tau_n+\theta} \int_{|r| \leq 1} r^2 \int_{\mathbb{S}(\mathbb{C}_n)} \|v_{ij}\|^4 \pi(dv) \rho_n^\alpha(dr) du \right] \\
(5.26) \quad & \leq \frac{1}{2} \|f''\|_\infty \delta \left(\int_{|r| \leq 1} r^2 \rho_n^\alpha(dr) \right) = C_1(f) \delta,
\end{aligned}$$

where we have used Proposition 4.2.3 in [17] and $C_1(f) > 0$ is a constant.

In order to estimate the second term in (5.25) we first estimate its integrand by using Taylor's theorem and Theorem 1.3 in [24]:

$$\begin{aligned}
& \left| f'(\lambda_m^{(n)}(X_{s-}^{(n)})) \left(\lambda_m^{(n)}(X_{s-} + r v v^*) - \lambda_m^{(n)}(X_{s-}) \right) - f'(\lambda_m^{(n)}(X_{s-}^{(n)})) \text{tr}(D\lambda_m^{(n)}(X_{s-}^{(n)}) r v v^*) \right| \\
& \leq \|f'\|_\infty \left| \lambda_m^{(n)}(X_{s-} + r v v^*) - \lambda_m^{(n)}(X_{s-}) - \text{tr}(D\lambda_m^{(n)}(X_{s-}^{(n)}) r v v^*) \right| \\
& \leq \|f'\|_\infty \frac{r^2}{2} \sum_{(r,l)(k,h)} \left| \frac{\partial^2}{\partial z_{rl} \partial z_{kh}} (\lambda_m^{(n)}(X_{s-}^{(n)})) \right| |(v v^*)_{rl}| |(v v^*)_{kh}| \\
& = \|f'\|_\infty r^2 \left[\sum_{k=1}^n \sum_{j \neq m} \frac{|\bar{u}_{km}^{(n)}(s) u_{kj}^{(n)}(s)|^2}{|\lambda_m^{(n)}(s) - \lambda_j^{(n)}(s)|} |(v v^*)_{kk}|^2 \right. \\
& \quad \left. + 2 \sum_{1 \leq k < h \leq n} \left(\sum_{j \neq m} \frac{|\bar{u}_{kj}^{(n)}(s) u_{hm}^{(n)}(s)|^2 + |\bar{u}_{hm}^{(n)}(s) u_{kj}^{(n)}(s)|^2}{|\lambda_m^{(n)}(s) - \lambda_j^{(n)}(s)|} \right) |(v v^*)_{kh}|^2 \right] \\
& \leq \|f'\|_\infty r^2 \sum_{j \neq m} \frac{1}{|\lambda_m^{(n)}(s) - \lambda_j^{(n)}(s)|} \left(\sum_{k=1}^n |\bar{u}_{km}^{(n)}(s) u_{kj}^{(n)}(s)|^2 |(v v^*)_{kk}|^2 \right. \\
& \quad \left. + 2 \sum_{1 \leq k < h \leq n} \left(|\bar{u}_{kj}^{(n)}(s) u_{hm}^{(n)}(s)|^2 + |\bar{u}_{hm}^{(n)}(s) u_{kj}^{(n)}(s)|^2 \right) |(v v^*)_{kh}|^2 \right),
\end{aligned}$$

where $u_{ij}(s)$, $i, j = 1, 2, \dots, n$ are the entries of a unitary random matrix. Now using Jensen's inequality and Lemma 1 for $p \in (1, 2)$ and $n > 1$,

$$\begin{aligned}
& \mathbb{E} \left[\left| \sum_{m=1}^n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| \leq 1} \left[f'(\lambda_m^{(n)}(X_{s-}^{(n)}))(\lambda_m^{(n)}(X_{s-} + r v v^*) - \lambda_m^{(n)}(X_{s-})) \right. \right. \right. \\
& \quad \left. \left. \left. - f'(\lambda_m^{(n)}(X_{s-}^{(n)})) \text{tr}(D\lambda_m^{(n)}(X_{s-}^{(n)}) r v v^*) \right] \rho_n^\alpha(dr) \pi(dv) ds \right| \right] \\
& \leq \mathbb{E} \left[\left| \sum_{m=1}^n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| \leq 1} \|f'\|_\infty r^2 \sum_{j \neq m} \frac{1}{|\lambda_m^{(n)}(s) - \lambda_j^{(n)}(s)|} \left(\sum_{k=1}^n |\bar{u}_{km}^{(n)}(s) u_{kj}^{(n)}(s)|^2 |(v v^*)_{kk}|^2 \right. \right. \right. \\
& \quad \left. \left. \left. + 2 \sum_{1 \leq k < h \leq n} (|\bar{u}_{kj}^{(n)}(s) u_{hm}^{(n)}(s)|^2 + |\bar{u}_{hm}^{(n)}(s) u_{kj}^{(n)}(s)|^2) |(v v^*)_{kh}|^2 \right) \rho_n^\alpha(dr) \pi(dv) ds \right| \right] \\
& \leq \|f'\|_\infty \left(\int_{|r| \leq 1} r^2 \rho_n^\alpha(dr) \right) \\
& \quad \times \mathbb{E} \left[\left| \sum_{m=1}^n \int_{\tau_n}^{\tau_n+\theta} \sum_{j \neq m} \frac{1}{|\lambda_m^{(n)}(s) - \lambda_j^{(n)}(s)|} \left[\sum_{k=1}^n |\bar{u}_{km}^{(n)}(s) u_{kj}^{(n)}(s)|^2 \int_{\mathbb{S}(\mathbb{C}_n)} |(v v^*)_{kk}|^2 \pi(dv) \right. \right. \right. \\
& \quad \left. \left. \left. + 2 \sum_{1 \leq k < h \leq n} (|\bar{u}_{kj}^{(n)}(s) u_{hm}^{(n)}(s)|^2 + |\bar{u}_{hm}^{(n)}(s) u_{kj}^{(n)}(s)|^2) \int_{\mathbb{S}(\mathbb{C}_n)} |(v v^*)_{kh}|^2 \pi(dv) \right] ds \right| \right] \\
& \leq C \frac{\|f'\|_\infty}{n^2} \left(\int_{|r| \leq 1} r^2 \rho_n^\alpha(dr) \right) \mathbb{E} \left[\left| \sum_{m=1}^n \sum_{j \neq m} \int_{\tau_n}^{\tau_n+\theta} \frac{1}{|\lambda_m^{(n)}(s) - \lambda_j^{(n)}(s)|} ds \right| \right] \\
& \leq C \frac{\|f'\|_\infty}{n^2} \left(\int_{|r| \leq 1} r^2 \rho_n^\alpha(dr) \right) \sum_{m=1}^n \sum_{j \neq m} \mathbb{E} \left[\theta^{p-1} \int_{\tau_n}^{\tau_n+\theta} \frac{1}{|\lambda_m^{(n)}(u) - \lambda_j^{(n)}(u)|^p} du \right]^{1/p} \\
& \leq C \|f'\|_\infty \left(\int_{|r| \leq 1} r^2 \rho_n^\alpha(dr) \right)^p \delta^{1-1/p} \int_0^{T'+\delta} \mathbb{E} \left[\frac{1}{|\lambda_m^{(n)}(u) - \lambda_j^{(n)}(u)|^p} du \right]^{1/p} \\
(5.27) \quad & \leq C K_p \|f'\|_\infty \left(\int_{|r| \leq 1} r^2 \rho_n^\alpha(dr) \right) \delta^{1-1/p} (T' + \delta)^{1/p} \leq \delta^{1-1/p} (T' + \delta)^{1/p} C_2(f),
\end{aligned}$$

for some generic constants $C, C_2(f) > 0$.

Now for the jump part in (5.21), we consider the remaining associated term and by an application of Lemma III.5 in [12] we obtain

$$\begin{aligned}
& \left| \sum_{m=1}^n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| > 1} \left[f(\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*)) - f(\lambda_m^{(n)}(X_{s-}^{(n)})) \right] \rho_n^\alpha(dr) \pi(dv) ds \right| \\
& \leq \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| > 1} (\|f\|'_1 \wedge |r| \|f'\|_\infty) \rho_n^\alpha(dr) \pi(dv) ds \\
(5.28) \quad & \leq \delta \int_{|r| > 1} (\|f\|'_1 \wedge |r| \|f'\|_\infty) \frac{1+r^2}{r^2} \rho(dr),
\end{aligned}$$

where, following Lemma III. 5 in [12], for any function Lipschitz g with finite total variation

$$\|g\|'_1 = \sup_{x_1 \leq y_1 \leq x_2 \leq \dots \leq y_n} \sum_{i=1}^n |g(y_i) - g(x_i)|,$$

and

$$\|g\|'_\infty = \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|}.$$

From (5.23), (5.24), (5.26), (5.27) and (5.28) we conclude that there exists a constant $K_1 > 0$, which does not depend on $n \geq 1$, such that

$$(5.29) \quad \sup_{n \geq 1} \sup_{\theta \in [0, \delta]} \mathbb{E} \left[\left| V_{\tau_n + \theta}^{n, f} - V_{\tau_n}^{n, f} \right| \right] < (\delta^2 \wedge \delta^{p-1}) K_1.$$

The next step is to prove the tightness of the laws of the martingale part of the semimartingale $\langle \mu^{(n)}, f \rangle$. Recall that the quadratic variation of the martingale $M^{n, f}$ is given by

$$\begin{aligned} \langle M^{n, f}, M^{n, f} \rangle_t &= \frac{\sigma_n^2}{n^3} \int_0^t \sum_{m=1}^n (f'(\lambda_m^{(n)}(X_{s-}^{(n)})))^2 ds \\ &\quad + \frac{1}{n} \int_0^t \int_{\mathbb{S}(\mathbb{C}_n)} \int_{\mathbb{R}} \left[\sum_{m=1}^n f(\lambda_m^{(n)}(X_{s-}^{(n)})) (\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*) - \lambda_m(X_{s-}^{(n)})) \right]^2 \rho_n^\alpha(dr) \pi(dv) ds. \end{aligned}$$

For the first term, note that

$$(5.30) \quad \frac{\sigma_n^2}{n^3} \mathbb{E} \left[\int_{\tau_n}^{\tau_n + \theta} \sum_{m=1}^n (f'(\lambda_m^{(n)}(X_{s-}^{(n)})))^2 ds \right] \leq \frac{(1 + \sigma)^2}{n^2} \|f'\|_\infty^2 \delta.$$

For the second term, similarly to the proof of Lemma 2, one obtains for $n \geq 1$

$$\begin{aligned} &\frac{1}{n} \mathbb{E} \left[\int_{\tau_n}^{\tau_n + \theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{\mathbb{R}} \left(\sum_{m=1}^n f(\lambda_m^{(n)}(X_{s-}^{(n)})) (\lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*) - \lambda_m(X_{s-}^{(n)})) \right)^2 \rho_n^\alpha(dr) \pi(dv) ds \right] \\ &\leq \frac{1}{n} \|f\|_\infty^2 \mathbb{E} \left[\int_{\tau_n}^{\tau_n + \theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{\mathbb{R}} \left(\sum_{m=1}^n \lambda_m^{(n)}(X_{s-}^{(n)} + r v v^*) - \lambda_m(X_{s-}^{(n)}) \right)^2 \rho_n^\alpha(dr) \pi(dv) ds \right] \\ (5.31) \quad &\leq \|f\|_\infty^2 \frac{\delta}{n} \int_{\mathbb{R}} r^2 \rho_n^\alpha(dr) \leq \delta \frac{n^{4\alpha}}{n(n^\alpha - 1)^2} C_3(f). \end{aligned}$$

for some constant $C_3(f) > 0$. From (5.30) and (5.31) there exists a constant $K_2 > 0$ independent of $n \geq 1$ such that

$$(5.32) \quad \sup_{n \geq 1} \sup_{\theta \in [0, \delta]} \mathbb{E} \left[\left| \langle M^{n, f}, M^{n, f} \rangle_{\tau_n + \theta} - \langle M^{n, f}, M^{n, f} \rangle_{\tau_n} \right| \right] < \delta K_2.$$

Let us fix $T > 0$. Then proceeding as in the first part of the proof, it can be seen that there exists a constant $K_1(T) > 0$ depending on T such that

$$\sup_{n \geq 1} \mathbb{E} \left[\left(\sup_{t \in [0, T]} V_t^{n, f} \right)^2 \right] < K_1(T).$$

On the other hand, from the proof of Lemma 2, there exists a constant $K_2(T) > 0$, that depends on T , such that

$$\sup_{n \geq 1} \mathbb{E} \left[\left(\sup_{t \in [0, T]} M_t^{n, f} \right)^2 \right] < K_2(T).$$

Therefore there exists a constant $K(T) > 0$ depending on T such that

$$(5.33) \quad \sup_{n \geq 1} \mathbb{E} \left[\left(\sup_{t \in [0, T]} \langle \mu_t^{(n)}, f \rangle \right)^2 \right] < K(T).$$

Now, from (5.29), (5.32) and (5.33), we can use the Aldous–Rebolledo criterion (see [2], [16], [26]) to conclude that the sequence of real processes $\{(\langle \mu_t^{(n)}, f \rangle)_{t \geq 0} : n \geq 1\}$ is tight, and that consequently the sequence of processes $\{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\}$ is tight in the space $\mathcal{D}(\mathbb{R}_+, \text{Pr}(\mathbb{R}))$, with $\text{Pr}(\mathbb{R})$ endowed with the topology of vague convergence.

It remains to extend the above result to the case when $\text{Pr}(\mathbb{R})$ is endowed with the topology of weak convergence. Note that taking $f = 1$, the sequence of real-valued processes $\{(\langle \mu_t^{(n)}, f \rangle)_{t \geq 0} : n \geq 1\}$ is tight. On the other hand note that Lemma 2 implies that for any convergent subsequence of $\{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\}$, the limit is strongly continuous. Therefore by an application of the Méléard–Roelly criterion (see [20]), it follows that the sequence $\{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\}$ is tight in the space $\mathcal{D}(\mathbb{R}_+, \text{Pr}(\mathbb{R}))$. \square

6. CHARACTERIZATION OF THE WEAK LIMIT OF THE MEASURE VALUED-PROCESSES

Let $z \in \mathbb{C} \setminus \mathbb{R}$ and $f_z(x) = (z - x)^{-1}$ for $x \in \mathbb{R}$. For any continuous function $(\mu_t)_{t \geq 0} \in C(\mathbb{R}_+, \text{Pr}(\mathbb{R}))$, define

$$\psi_\mu(t, z) := \int_{\mathbb{R}} f_z(x) \mu_t(dx).$$

We identify the weak limit of the sequence $\{(\langle \mu_t^{(n)}, f \rangle)_{t \geq 0} : n \geq 1\}$ to be the family $(\mu_t)_{t \geq 0}$ that satisfies the Burger’s equation of the law of the free Lévy process $\{Z_t : t \geq 0\}$, that appears in the proof of Theorem 5.10 in Bercovici and Voiculescu [7], for free infinitely divisible distributions with possibly unbounded support.

Theorem 2. *Assume that μ_0^n converges weakly to δ_0 . Then the family of measure-valued processes $\{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\}$ converges weakly in $\mathcal{D}(\mathbb{R}_+, \text{Pr}(\mathbb{R}))$ to a unique continuous probability-measure valued function $(\mu_t)_{t \geq 0}$, satisfying for each $t \geq 0$,*

$$\frac{\partial}{\partial t} \psi_\mu(t, z) = -\sigma^2 \psi_\mu(t, z) \frac{\partial}{\partial z} \psi_\mu(t, z) - \eta \frac{\partial}{\partial z} \psi_\mu(t, z) - \frac{\partial}{\partial z} \psi_\mu(t, z) \int_{\mathbb{R} \setminus \{0\}} \frac{\psi_\mu(t, z) + r}{1 - r \psi_\mu(t, z)} \rho(dr).$$

The following two auxiliary lemmas are useful for identifying the limit law of the sequence of empirical measures $\{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\}$. Their proofs are based on ideas from the proof of Lemma III.6 in [12].

Lemma 3. *Assume that (v_1, v_2, \dots, v_n) is a random vector Haar distributed on the unit complex sphere, and that x_1, \dots, x_n are fixed real numbers. Then*

$$\lim_{n \rightarrow \infty} \left| t \frac{\sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2}}{1 - t \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)}} - \frac{\frac{t}{n} \sum_{i=1}^n \frac{1}{(\zeta - x_i)^2}}{1 - \frac{t}{n} \sum_{i=1}^n \frac{1}{(\zeta - x_i)}} \right| = 0, \quad \text{in probability,}$$

for $\zeta \in \mathbb{C} \setminus \mathbb{R}$, $t > 0$.

Proof. Define the function $\beta : \mathbb{C}^n \rightarrow \mathbb{R}$ by

$$\beta(v_1, \dots, v_n) = t \frac{\sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2}}{1 - t \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)}}.$$

Let $\varepsilon > 0$. From the proof of Lemma III.6 in [12], there exists a constant $\tilde{C} > 0$ such that

$$(6.34) \quad \mathbb{P}(|\beta(v_1, \dots, v_n) - \mathbb{E}[\beta(v_1, \dots, v_n)]| > \varepsilon) \leq \tilde{C} \exp\left(-\frac{(n-1)}{2\|\beta\|_\infty^2} \varepsilon^2\right),$$

where we have used the fact that β is Lipschitz continuous with Lipschitz constant $\|\beta\|'_\infty := \sup_{v \neq v'} \left| \frac{\beta(v) - \beta(v')}{|v - v'|} \right| < \infty$. Next we prove that β is Lipschitz. Let $v = (v_1, \dots, v_n)$ and $v' = (v'_1, \dots, v'_n)$ be two vectors on the complex unit sphere. Then,

$$\begin{aligned} |\beta(v_1, \dots, v_n) - \beta(v'_1, \dots, v'_n)| &= \left| t \frac{\sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2}}{1 - t \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)}} - t \frac{\sum_{i=1}^n \frac{|v'_i|^2}{(\zeta - x_i)^2}}{1 - t \sum_{i=1}^n \frac{|v'_i|^2}{(\zeta - x_i)}} \right| \\ &\leq \frac{1}{A} \left| \sum_{i=1}^n \frac{|v_i|^2 - |v'_i|^2}{(\zeta - x_i)^2} \right| + \frac{1}{AA'} \left| \sum_{i=1}^n \frac{|v'_i|^2}{(\zeta - x_i)^2} \right| \left| \frac{\sum_{i=1}^n |v_i|^2 - |v'_i|^2}{(\zeta - x_i)} \right|, \end{aligned}$$

where

$$A = \left| 1 - t \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)} \right| \quad \text{and} \quad A' = \left| 1 - t \sum_{i=1}^n \frac{|v'_i|^2}{(\zeta - x_i)} \right|.$$

Since $\zeta \in \mathbb{C} \setminus \mathbb{R}$, there exists a constant $K > 0$ such that

$$\frac{1}{|\zeta - x|} < K, \quad \text{for all } x \in \mathbb{R},$$

which implies that

$$\left| \sum_{i=1}^n \frac{|v'_i|^2}{(\zeta - x_i)^2} \right| \leq K^2 \sum_{i=1}^n |v'_i|^2 = K^2.$$

Similarly, there exists a constant $\tilde{K} > 0$ such that

$$\left| \frac{1}{t} - \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x)} \right| \wedge \left| \frac{1}{t} - \sum_{i=1}^n \frac{|v'_i|^2}{(\zeta - x)} \right| > \tilde{K} \quad \text{for all } x \in \mathbb{R}, t > 0.$$

Therefore

$$\begin{aligned} |\beta(v_1, \dots, v_n) - \beta(v'_1, \dots, v'_n)| &\leq \frac{1}{\tilde{K}} \left| \sum_{i=1}^n \frac{|v_i|^2 - |v'_i|^2}{(\zeta - x_i)^2} \right| + \frac{K^2}{\tilde{K}^2} \left| \sum_{i=1}^n \frac{|v_i|^2 - |v'_i|^2}{(\zeta - x_i)} \right| \\ &\leq 2K^2 \left(\frac{1}{\tilde{K}} + \frac{K}{\tilde{K}^2} \right) \sum_{i=1}^n |v_i - v'_i| \leq C \|v - v'\|, \end{aligned}$$

where $C > 0$ is a constant that does not depend on either $n \geq 1$ or x_1, \dots, x_n . Hence β is Lipschitz with $\sup_{v \neq v'} \left| \frac{\beta(v) - \beta(v')}{|v - v'|} \right| \leq C$.

On the other hand, note that

$$\begin{aligned}
 & \left| \mathbb{E} \beta(v_1, \dots, v_n) - \frac{\mathbb{E} \left[t \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} \right]}{\mathbb{E} \left[1 - t \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} \right]} \right| \\
 & \leq \tilde{K} \mathbb{E} \left(\left| \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} - \mathbb{E} \left[\sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} \right] \right| \right) + \\
 (6.35) \quad & \tilde{K}^2 \mathbb{E} \left| \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} \right| \mathbb{E} \left| \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)} - \mathbb{E} \left[\sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)} \right] \right|.
 \end{aligned}$$

Next we prove that both terms on the right hand side go to zero as $n \rightarrow \infty$. Note that the function $\tilde{\beta} : \mathbb{C}^n \rightarrow \mathbb{R}$ defined by

$$\tilde{\beta}(v_1, \dots, v_n) = \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2}.$$

is Lipschitz continuous since for v and v' in the complex unit sphere,

$$\left| \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} - \sum_{i=1}^n \frac{|v'_i|^2}{(\zeta - x_i)^2} \right| \leq 2K^2 \|v - v'\|,$$

with Lipschitz constant $\|\tilde{\beta}\|'_\infty \leq 2K^2$. Now from the proof of Lemma III.6 in [12] there exists a constant $\tilde{C}_1 > 0$ such that

$$\mathbb{P} \left(\left| \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} - \mathbb{E} \left[\sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} \right] \right| > \varepsilon \right) \leq \tilde{C}_1 \exp \left(-\frac{(n-1)}{2\|\tilde{\beta}\|_\infty'^2} \varepsilon^2 \right),$$

and therefore

$$\begin{aligned}
 \mathbb{E} \left(\left| \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} - \mathbb{E} \left[\sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} \right] \right| \right) & \leq 2K \mathbb{P} \left(\left| \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} - \mathbb{E} \left[\sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} \right] \right| > \varepsilon \right) + \varepsilon \\
 & \leq 2K \tilde{C}_1 \exp \left(-\frac{(n-1)}{2\|\tilde{\beta}\|_\infty'^2} \varepsilon^2 \right) + \varepsilon.
 \end{aligned}$$

Letting $\varepsilon \downarrow 0$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\left| \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} - \mathbb{E} \left[\sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)^2} \right] \right| \right) = 0,$$

uniformly for any choice of real numbers x_1, \dots, x_n . Similarly, it can be proved that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\left| \sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)} - \mathbb{E} \left[\sum_{i=1}^n \frac{|v_i|^2}{(\zeta - x_i)} \right] \right| \right) = 0,$$

uniformly for any choice of real numbers x_1, \dots, x_n .

From (6.34) and (6.35),

$$\lim_{n \rightarrow \infty} \beta(v_1, \dots, v_n) - t \frac{\frac{1}{n} \sum_{i=1}^n \frac{1}{(\zeta - x_i)^2}}{1 - \frac{t}{n} \sum_{i=1}^n \frac{1}{(\zeta - x_i)}} = 0,$$

in probability, uniformly for any choice of real numbers x_1, \dots, x_n . \square

Lemma 4. *There exists a subsequence $\{n_k\}_{k \geq 1}$ such that*

$$\lim_{k \rightarrow \infty} \sum_{m=1}^{n_k} \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) \bar{u}_{mi}^{(n_k)}(s-) u_{mj}^{(n_k)}(s-) v_i \bar{v}_j - \int_{\mathbb{R}} f'_z(x) \mu_{s-}^{(n_k)}(dx) = 0 \quad \text{a.s.},$$

where (v_1, v_2, \dots, v_n) is a Haar distributed random vector on the unit sphere of \mathbb{C}^n and $(u_{ij}^{(n)})$ is an $n \times n$ unitary random matrix.

Proof. We split the above sum into two parts: the first is the sum of all terms with $i = j$, and the second has all terms satisfying $i \neq j$. For the terms with $i = j$, we consider for fixed $\lambda := (\lambda_1, \dots, \lambda_n)$ and $u := (u_{mi})_{m,i=1}^n$ the function $\beta : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$\beta(v_1, \dots, v_n) := \sum_{m=1}^n \sum_{i=1}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) |\bar{u}_{mi}^{(n)}(s-)|^2 |v_i|^2.$$

The function β is Lipschitz continuous in (v_1, \dots, v_n) since

$$\begin{aligned} |\beta(v_1, \dots, v_n) - \beta(v'_1, \dots, v'_n)| &= \left| \sum_{m=1}^n \sum_{i=1}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) |\bar{u}_{mi}^{(n)}(s-)|^2 |v_i - v'_i| |v_i + v'_i| \right| \\ &\leq 2 \|f'_z\|_{\infty} \sum_{i=1}^n |v_i - v'_i| \leq 2 \|f'_z\|_{\infty} \left(\sum_{i=1}^n |v_i - v'_i|^2 \right)^{1/2}. \end{aligned}$$

Hence we can apply to the Lipschitz function β the concentration result in the proof of Lemma III. 6 in [12] to obtain

$$\begin{aligned} \mathbb{P} \left[\left(\sum_{m=1}^n \sum_{i=1}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) |\bar{u}_{mi}^{(n)}(s-)|^2 |v_i|^2 - \mathbb{E} \left(\sum_{m=1}^n \sum_{i=1}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) |\bar{u}_{mi}^{(n)}(s-)|^2 |v_i|^2 \middle| \lambda, u \right) \right) > \varepsilon \right] \\ \leq C \exp \left(-\frac{n-1}{8 \|f'_z\|_{\infty}^2} \varepsilon^2 \right), \end{aligned}$$

where $\mathbb{E}(\cdot | \lambda, u)$ denotes the conditional expectation with respect to λ and u . This implies that

$$(6.36) \quad \sum_{m=1}^n \sum_{i=1}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) |\bar{u}_{mi}^{(n)}(s-)|^2 |v_i|^2 - \mathbb{E} \left(\sum_{m=1}^n \sum_{i=1}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) |\bar{u}_{mi}^{(n)}(s-)|^2 |v_i|^2 \middle| \lambda, u \right) \rightarrow 0,$$

in probability as $n \rightarrow \infty$ and note that

$$\begin{aligned} \mathbb{E} \left(\sum_{m=1}^n \sum_{i=1}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) |\bar{u}_{mi}^{(n)}(s-)|^2 |v_i|^2 \middle| \lambda, u \right) &= \frac{1}{n} \sum_{m=1}^n \sum_{i=1}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) |\bar{u}_{mi}^{(n)}(s-)|^2 \\ &= \frac{1}{n} \sum_{m=1}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) = \int_{\mathbb{R}} f'_z(x) \mu_{s-}^{(n_k)}(dx). \end{aligned}$$

Therefore, from (6.36), there exists a subsequence $\{n_k\}_{k \geq 1}$ such that

$$(6.37) \quad \lim_{k \rightarrow \infty} \sum_{m=1}^{n_k} \sum_{i=1}^{n_k} f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) |\bar{u}_{mi}^{(n_k)}(s-)|^2 |v_i|^2 - \int_{\mathbb{R}} f'_z(x) \mu_{s-}^{(n_k)}(dx) = 0 \quad \text{a.s.}$$

Now for the terms containing indices $i \neq j$, note that

$$\begin{aligned}
& \mathbb{E} \left(\left| \sum_{m=1}^n \sum_{i \neq j}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) \bar{u}_{mi}^{(n)}(s-) u_{mj}^{(n)}(s-) v_i \bar{v}_j \right|^2 \right) \\
&= \mathbb{E} \left(\sum_{m=1}^n \sum_{i \neq j}^n \sum_{m'=1}^n \sum_{i' \neq j'}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) \bar{u}_{mi}^{(n)}(s-) u_{mj}^{(n)}(s-) v_i \bar{v}_j \bar{f}'_z(\lambda_{m'}^{(n)}(X_{s-}^{(n)})) \bar{u}_{m'i'}^{(n)}(s-) u_{m'j'}^{(n)}(s-) v_{j'} \bar{v}_{i'} \right) \\
&= \mathbb{E} \left(\sum_{m=1}^n \sum_{i \neq j}^n \sum_{m'=1}^n \sum_{i' \neq j'}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) \bar{u}_{mi}^{(n)}(s-) u_{mj}^{(n)}(s-) \right. \\
&\quad \left. \times \bar{f}'_z(\lambda_{m'}^{(n)}(X_{s-}^{(n)})) \bar{u}_{m'j'}^{(n)}(s-) u_{m'i'}^{(n)}(s-) \mathbb{E} \left[v_i \bar{v}_j v_{j'} \bar{v}_{i'} \mid \lambda, u \right] \right).
\end{aligned}
\tag{6.38}$$

Since (v_1, \dots, v_n) is a Haar distributed random vector in the unit sphere of \mathbb{C}^n from Proposition 4.2.3 in [17],

$$\mathbb{E} (v_i \bar{v}_j v_{j'} \bar{v}_{i'}) = \begin{cases} \frac{1}{n^2-1} & \text{if } i = i' \text{ and } j = j', \\ 0 & \text{otherwise,} \end{cases}$$

and therefore (6.38) turns out to be

$$\begin{aligned}
&= \frac{1}{n^2-1} \mathbb{E} \left(\sum_{m=1}^n \sum_{m'=1}^n \sum_{i=1}^n \sum_{j=1}^n f'_z(\lambda_m^{(n)}(X_{s-}^{(n)})) \bar{f}'_z(\lambda_{m'}^{(n)}(X_{s-}^{(n)})) \bar{u}_{mi}^{(n)}(s-) u_{m'i}^{(n)}(s-) \bar{u}_{m'j}^{(n)}(s-) u_{mj}^{(n)}(s-) \right) \\
&= \frac{1}{n^2-1} \mathbb{E} \left(\sum_{m=1}^n \sum_{i=1}^n \sum_{j=1}^n |f'_z(\lambda_m^{(n)}(X_{s-}^{(n)}))|^2 |u_{mi}^{(n)}(s-)|^2 |u_{mj}^{(n)}(s-)|^2 \right) \\
&\leq \frac{1}{n^2-1} \|f'_z\|_\infty^2 \mathbb{E} \left(\sum_{m=1}^n \sum_{i=1}^n \sum_{j=1}^n |u_{mi}^{(n)}(s-)|^2 |u_{mj}^{(n)}(s-)|^2 \right) = \frac{n}{n^2-1} \|f'_z\|_\infty^2,
\end{aligned}$$

where the last term converges to 0 as $n \rightarrow \infty$.

Therefore there exists a subsequence which we also denote, without loss of generality, by $\{n_k\}_{k \geq 1}$ such that

$$\lim_{k \rightarrow \infty} \sum_{m=1}^{n_k} \sum_{i \neq j}^{n_k} f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) \bar{u}_{mi}^{(n_k)}(s-) u_{mj}^{(n_k)}(s-) v_i \bar{v}_j = 0 \quad \text{a.s.}
\tag{6.39}$$

Finally, the assertion follows from (6.37) and (6.39). □

Proof of Theorem 2. For $\mu_t^{(n)}$, put $\psi^n(t, z) := \int_{\mathbb{R}} f_z(x) \mu_t^{(n)}(dx)$. By (4.19)

$$\begin{aligned}
 \psi^n(t, z) &= \psi^n(0, z) - \sigma_n^2 \int_0^t \psi^n(s, z) \frac{\partial}{\partial z} \psi^n(s, z) ds - \gamma \int_0^t \frac{\partial}{\partial z} \psi^n(s, z) ds + M_t^{n,f} \\
 &+ \sum_{m=1}^n \int_0^t \int_{\mathbb{S}(\mathbb{C}_n)} \int_{\mathbb{R}} \left[f_z(\lambda_m^{(n)}(X_{u-}^{(n)} + r v v^*)) - f_z(\lambda_m^{(n)}(X_{u-}^{(n)})) \right. \\
 &\quad \left. - f'_z(\lambda_m^{(n)}(X_{u-}^{(n)})) \text{tr}(D\lambda_m^{(n)}(X_{u-}^{(n)}) r v v^*) 1_{\{|r| \leq 1\}} \right] \rho_n^\alpha(dr) \pi(dv) du.
 \end{aligned}
 \tag{6.40}$$

We shall prove the convergence of each term in the above equation for a certain subsequence.

From Proposition 3, the family $\{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\}$ is relatively compact. Let $\{(\mu_t^{(n_k)})_{t \geq 0} : k \geq 1\}$ be a subsequence that converges weakly to $(\mu_t)_{t \geq 0}$. Then

$$\lim_{k \rightarrow \infty} \psi^{n_k}(t, z) = \lim_{k \rightarrow \infty} \int_{\mathbb{R}} f_z(x) \mu_t^{(n_k)}(dx) = \int_{\mathbb{R}} f_z(x) \mu_t(dx) = \psi_\mu(t, z),
 \tag{6.41}$$

and similarly

$$\begin{aligned}
 \lim_{k \rightarrow \infty} \int_0^t \psi^{n_k}(s, z) \frac{\partial}{\partial z} \psi^{n_k}(s, z) ds &= - \lim_{k \rightarrow \infty} \int_0^t \int_{\mathbb{R}^2} \frac{\mu_s^{n_k}(dx) \mu_s^{n_k}(dy)}{(z-x)(z-y)^2} ds \\
 &= - \int_0^t \int_{\mathbb{R}^2} \frac{\mu_s(dx) \mu_s(dy)}{(z-x)(z-y)^2} ds = \int_0^t \psi_\mu(s, z) \frac{\partial}{\partial z} \psi_\mu(s, z) ds,
 \end{aligned}
 \tag{6.42}$$

and for the drift term

$$\lim_{k \rightarrow \infty} \int_0^t \frac{\partial}{\partial z} \psi^{n_k}(s, z) ds = - \lim_{k \rightarrow \infty} \int_0^t \int_{\mathbb{R}} \frac{\mu_s^{n_k}(dx)}{(z-x)^2} ds = - \int_0^t \int_{\mathbb{R}} \frac{\mu_s(dx)}{(z-x)^2} ds = \int_0^t \frac{\partial}{\partial z} \psi_\mu(s, z) ds.
 \tag{6.43}$$

Next, by Lemma III.7 in [12], we have for $v = (v_1, \dots, v_{n_k})$ that

$$\sum_{m=1}^{n_k} \left(f_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)} + r v v^*)) - f_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) \right) = r \frac{\sum_{i=1}^{n_k} \frac{|v_i|^2}{(z - \lambda_i^{(n_k)}(s-))^2}}{1 - r \sum_{i=1}^{n_k} \frac{|v_i|^2}{(z - \lambda_i^{(n_k)}(s-))}},
 \tag{6.44}$$

and now, using Lemma 3 and choosing a suitable subsequence, which without loss of generality we also denote by $\{\mu_t^{(n_k)}\}_{t \geq 0}$, we have that

$$\left| r \frac{\sum_{i=1}^{n_k} \frac{|v_i|^2}{(z - \lambda_i^{(n_k)}(s-))^2}}{1 - r \sum_{i=1}^{n_k} \frac{|v_i|^2}{(z - \lambda_i^{(n_k)}(s-))}} - r \frac{\int_{\mathbb{R}} \frac{1}{(z-x)^2} \mu_{s-}(dx)}{1 - r \int_{\mathbb{R}} \frac{1}{(z-x)} \mu_{s-}(dx)} \right|$$

$$\leq \left| r \frac{\sum_{i=1}^{n_k} \frac{|v_i|^2}{(z - \lambda_i^{(n_k)}(s-))^2}}{1 - r \sum_{i=1}^{n_k} \frac{|v_i|^2}{(z - \lambda_i^{(n_k)}(s-))}} - r \frac{\frac{1}{n_k} \sum_{i=1}^{n_k} \frac{1}{(z - \lambda_i^{(n_k)}(s-))^2}}{1 - \frac{r}{n_k} \sum_{i=1}^{n_k} \frac{1}{(z - \lambda_i^{(n_k)}(s-))}} \right|$$

$$+ \left| r \frac{\int_{\mathbb{R}} \frac{1}{(z-x)^2} \mu_{s-}^{(n_k)}(dx)}{1 - r \int_{\mathbb{R}} \frac{1}{(z-x)} \mu_{s-}^{(n_k)}(dx)} - r \frac{\int_{\mathbb{R}} \frac{1}{(z-x)^2} \mu_{s-}(dx)}{1 - r \int_{\mathbb{R}} \frac{1}{(z-x)} \mu_{s-}(dx)} \right|.$$

Note that both terms on the right hand side of the above inequality converge a.s. to 0 as $k \rightarrow \infty$. Therefore,

$$(6.45) \quad \lim_{k \rightarrow \infty} r \frac{\sum_{i=1}^{n_k} \frac{|v_i|^2}{(z - \lambda_i^{(n_k)}(s-))^2}}{1 - r \sum_{i=1}^{n_k} \frac{|v_i|^2}{(z - \lambda_i^{(n_k)}(s-))}} = r \frac{\int_{\mathbb{R}} \frac{1}{(z-x)^2} \mu_{s-}(dx)}{1 - r \int_{\mathbb{R}} \frac{1}{(z-x)} \mu_{s-}(dx)} = \frac{-r \frac{\partial}{\partial z} \psi_{\mu}(s-, z)}{1 - r \psi_{\mu}(s-, z)}.$$

On the other hand, using (2.8) we have that

$$(6.46) \quad \sum_{m=1}^{n_k} f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) \text{tr}(D\lambda_m^{(n_k)}(X_{s-}^{(n_k)}) r v v^*) = r \sum_{m=1}^{n_k} \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) |\bar{u}_{mi}^{(n_k)}(s-) u_{mj}^{(n_k)}(s-)| (v v^*)_{ji}$$

$$= r \sum_{m=1}^{n_k} \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) |\bar{u}_{mi}^{(n_k)}(s-) u_{mj}^{(n_k)}(s-)| v_i \bar{v}_j.$$

Now from Lemma 4 we have

$$\lim_{k \rightarrow \infty} \left| r \sum_{m=1}^{n_k} \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) |\bar{u}_{mi}^{(n_k)}(s-) u_{mj}^{(n_k)}(s-)| v_i \bar{v}_j - \frac{r}{n_k} \sum_{m=1}^{n_k} f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) \right| = 0, \quad \text{a.s.}$$

and therefore

$$\lim_{k \rightarrow \infty} \left| r \sum_{m=1}^{n_k} \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) |\bar{u}_{mi}^{(n_k)}(s-) u_{mj}^{(n_k)}(s-)| v_i \bar{v}_j - r \int_{\mathbb{R}} f'_z(x) \mu_{s-}(dx) \right|$$

$$\leq \lim_{k \rightarrow \infty} \left| r \sum_{m=1}^{n_k} \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) |\bar{u}_{mi}^{(n_k)}(s-) u_{mj}^{(n_k)}(s-)| v_i \bar{v}_j - \frac{r}{n_k} \sum_{m=1}^{n_k} f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) \right|$$

$$+ \lim_{k \rightarrow \infty} \left| \int_{\mathbb{R}} f'_z(x) \mu_{s-}^{(n_k)}(dx) - \int_{\mathbb{R}} f'_z(x) \mu_{s-}(dx) \right| = 0, \quad \text{a.s.}$$

So it has been proved that

$$(6.47) \quad \lim_{k \rightarrow \infty} r \sum_{m=1}^{n_k} \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) |\bar{u}_{mi}^{(n_k)}(s-) u_{mj}^{(n_k)}(s-)| v_i \bar{v}_j = -r \frac{\partial}{\partial z} \psi_{\mu}(s-, z), \quad \text{a.s.}$$

Now from (6.44), (6.45), (6.46), (6.47) and the dominated convergence theorem,

$$\begin{aligned}
 & \lim_{k \rightarrow \infty} \sum_{m=1}^{n_k} \int_0^t \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| \leq 1} \left[f_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)} + r v v^*)) \right. \\
 & \quad \left. - f_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) - f'_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) \text{tr}(D\lambda_m^{(n_k)}(X_{s-}^{(n_k)}) r v v^*) \right] \rho_n^\alpha(dr) \pi(dv) ds \\
 (6.48) \quad & = \int_0^t \int_{|r| \leq 1} \frac{-r^2 \psi(s, z)}{1 - r \psi(s, z)} \frac{\partial}{\partial z} \psi_\mu(s, z) \tilde{\rho}(dr) ds.
 \end{aligned}$$

where

$$\tilde{\rho}(dr) = \frac{1 + r^2}{r^2} [\rho(-dr) + \rho(dr)] 1_{(0, \infty)}(r).$$

Similarly, from (6.44), (6.45) and the dominated convergence theorem,

$$\begin{aligned}
 & \lim_{k \rightarrow \infty} \sum_{m=1}^{n_k} \int_0^t \int_{\mathbb{S}(\mathbb{C}_n)} \int_{|r| > 1} \left[f_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)} + r v v^*)) - f_z(\lambda_m^{(n_k)}(X_{s-}^{(n_k)})) \right] \rho_n^\alpha(dr) \pi(dv) ds \\
 (6.49) \quad & = \int_0^t \int_{|r| > 1} \frac{r}{1 - r \psi(s, z)} \frac{\partial}{\partial z} \psi_\mu(s, z) \tilde{\rho}(dr) ds.
 \end{aligned}$$

Finally, from (6.41), (6.42), (6.43), (6.48), (6.49), Lemma 2, and using (3.13) and (3.14), the following SDE for the limit of the spectral measure-valued process of $X_t^{(n)}$ can be derived:

$$\begin{aligned}
 \frac{\partial}{\partial t} \psi_\mu(t, z) &= -\sigma^2 \psi_\mu(t, z) \frac{\partial}{\partial z} \psi_\mu(t, z) - \gamma \frac{\partial}{\partial z} \psi_\mu(t, z) - \frac{\partial}{\partial z} \psi_\mu(t, z) \int_{|r| \leq 1} \frac{\psi_\mu(t, z)}{1 - r \psi_\mu(t, z)} r^2 \tilde{\rho}(dr), \\
 & - \frac{\partial}{\partial z} \psi_\mu(s, z) \int_{|r| > 1} \frac{r}{1 - r \psi_\mu(s, z)} \tilde{\rho}(dr) \\
 &= -\sigma^2 \psi_\mu(t, z) \frac{\partial}{\partial z} \psi_\mu(t, z) - \eta \frac{\partial}{\partial z} \psi_\mu(t, z) - \frac{\partial}{\partial z} \psi_\mu(s, z) \left(\int_{|r| \leq 1} \frac{\psi_\mu(t, z)}{1 - r \psi_\mu(t, z)} r^2 \tilde{\rho}(dr) + \int_{|r| \leq 1} r \rho(dr) \right) \\
 & - \frac{\partial}{\partial z} \psi_\mu(s, z) \left(\int_{|r| > 1} \frac{r}{1 - r \psi_\mu(s, z)} \tilde{\rho}(dr) - \int_{|r| > 1} \frac{1}{r} \rho(dr) \right) \\
 &= -\sigma^2 \psi_\mu(t, z) \frac{\partial}{\partial z} \psi_\mu(t, z) - \eta \frac{\partial}{\partial z} \psi_\mu(t, z) - \frac{\partial}{\partial z} \psi_\mu(t, z) \int_{\mathbb{R} \setminus \{0\}} \frac{\psi_\mu(t, z) + r}{1 - r \psi_\mu(t, z)} \rho(dr),
 \end{aligned}$$

which is the Burger equation that appears in the proof of Theorem 5.10 in [7] (see also [17, Lemma 3.3.9]). \square

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